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Relativistic Mass Distribution in Event–Anti-event System and "Realistic" Equation of State for Hot Hadronic Matter

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Abstract

We find the equation of state $p, \rho \propto T^6$, which gives the value of the sound velocity $c^2 = 0.20$, in agreement with the "realistic" equation of state for hot hadronic matter suggested by Shuryak, in the framework of a covariant relativistic statistical mechanics of an event–anti-event system with small chemical and mass potentials. The relativistic mass distribution for such a system is obtained and shown to be a good candidate for fitting hadronic resonances, in agreement with the phenomenological models of Hagedorn, Shuryak, *et al.* This distribution provides a correction to the value of specific heat 3/2, of the order of 5.5%, at low temperatures.

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1   Introduction

One of the main goals of experiments with high-energy nuclear collisions is to produce and to study hadronic matter, in particular, trying to reach conditions at which the phase transitions into the quark-gluon plasma phase can take place [1]-[5]. The physics of hot hadronic matter has not been studied much, although such matter is already produced in current experiments in high-energy physics.

The experimental data on multiple hadron production obtained in recent years are in agreement with the main consequences of the theory formulated by Landau [6] over 40 years ago. However, with a sufficiently quantitative approach, it becomes necessary to consider a number of physical effects which bring about certain modifications of the results obtained in the fundamental work [6]. For instance, the solution of the equations of motion obtained by Landau differs from a numerical calculation of Melekhin [7], as a result of an inaccurate estimate. A more accurate analytic solution was given in [8].

The equation of state in Landau's work is taken to be $p = \rho/3$ (where p is the pressure and ρ is the energy density), corresponding to an ultra-relativistic gas. However, the lab energies $s \equiv (p_1+p_2)^2 \stackrel{<}{\sim} 3 \times 10^3$ GeV^2 correspond to initial temperatures $T \stackrel{<}{\sim} 1$ GeV. In this temperature range, the interaction of the hadrons is strong and has mainly a resonant character, the masses of the resonances being comparable with the temperature. Thus, hadronic matter under these conditions is neither an ideal nor an ultra-relativistic gas.

Corrections in the equation of state due to the interaction of the hadrons have been discussed in the literature for the last three decades [1],[4],[8]-[14]. These considerations were based mostly on a phenomenological model in which the Landau theory is applied to all particles except the leading ones, i.e., the fragments of the initial particles, whose characteristics are taken directly from experimental data. The framework for the latter considerations is the QCD phase transition of hadronic matter into the quark-gluon plasma.

Hot hadronic matter, in which volumes per particle are a few cubic fermis, is certainly made out of individual hadrons. It is clear that, at low temperatures $T << m_\pi$, one has a very rare (and therefore ideal) gas of the lightest hadrons, the pions. As the temperature is raised and the gas becomes more dense, one should take into account interactions among the particles. This was done using the following three approaches: (i) the low-temperature expansion, (ii) the resonance gas, (iii) the quasi-particle gas.

The first approach for the pion gas is based on the Weinberg theory of pion inter-actions [15], which uses the non-linear Lagrangian containing all processes quadratic in pion momentum. Its first application to calculation of the thermodynamic param-eters of the pion gas was done in ref. [1]. One of the important consequences of this work was that, after isospin averaging, all corrections quadratic in momenta cancel each other, and corrections proportional to the pion mass (Weinberg $\pi\pi$ scattering lengths) are nearly compensated, producing negligible corrections at the 1% level. Corrections of the second order in the Weinberg Lagrangian were also estimated in [1], and a much more systematic study of the problem including quartic terms in the mesonic Lagrangian were made in ref. [16]. Without going into discussion of these works we remark that the applicability of this approach is limited by temperatures $T < 100$ MeV, for which typical collision energies are significantly below resonance .However, such T are lower than even the lowest temperature available in experiments, because the so-called break-up temperatures are typically $T \simeq 120 - 150$ MeV.

The idea of resonance gas was first suggested by Belenky and Landau [17], as early as 1956. They used the Beth-Uhlenbeck formula [18], well known in the theory of the non-ideal gases [19], which relates the second virial coefficient with scattering phase shifts $\delta_l(p)$. In particular, in this approximation the "internal" part of the statistical sum can be written as a sum over bound states plus the scattering part[1]:

$$Z_{\text{int}} = \sum_n e^{-E_n/T} + \frac{1}{\pi} \sum_l \int dp \frac{d\delta_l(p)}{dp} e^{-p^2/2mT}. \tag{1.1}$$

Using Eq. (1.1) one can prove that a narrow and elastic resonance is analogous to adding one extra physical state to the system.

This observation was later used by many authors, in particular, by Hagedorn [9], who has generalized this statement to the idea that all hadrons (both stable and resonances) should be treated as real degrees of freedom of hadronic matter.The statistical bootstrap model [9],[10] initiated by Hagedorn, which requires that the spectra of the resonances and of the system as a whole coincide, provides a resonance spectrum

$$\rho(m) \sim m^a \exp(m/T_0), \tag{1.2}$$

where a and T_0 are some parameters. The statistical sum diverges when $T > T_0$, which indicates that the theory involves a limiting temperature (the so-called "hadronic boiling point" [9]) to which the system can be heated. Such behavior indicates that there is some phase transition, and that the language used is meaningless above it. This phase transition, as well as thermodynamic consequences of the mass spectrum (1.2), were discussed in ref. [20]. The expression (1.2) with $a = -5/2$ is in good agreement with experiment in the resonance mass region $m \lesssim 1.2$ GeV [8].

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[1]This formula is written down for the nonrelativistic case, as it appears in textbooks on statistical mechanics, e.g., in [19].

In 1972, Shuryak [8], in place of (1.2), used the simple power parametrization

$$\rho(m) \sim m^k, \tag{1.3}$$

which also describes experiment in this mass region for $k \approx 3$. The energy density is expressed by [8] ($E^2 = \mathbf{p}^2 + m^2$)

$$\rho = \sum_\sigma \int dm \rho(m) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{E}{\exp(\frac{E}{T}) + \sigma}, \quad \sigma = \pm 1. \tag{1.4}$$

Since $\rho(m)$ is a rapidly growing function, the main contribution to integrals of type (1.4) is given by the mass region in which $\exp(m/T) >> 1$, and the difference in properties of bosons and fermions is irrelevant. Upon substitution of (1.3) into (1.4) one finds that the energy density and, analogously, the pressure, defined by

$$p = \frac{1}{3} \sum_\sigma \int dm \rho(m) \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\mathbf{p}^2}{E} \frac{1}{\exp(\frac{E}{T}) + \sigma}, \tag{1.5}$$

is proportional to T^{k+5}. Therefore, the velocity of sound c, defined by

$$c^2 = \frac{dp}{d\rho}, \tag{1.6}$$

turns out to be a temperature independent constant, and the equation of state belongs to the class considered by Melekhin [21]. For this case, the expressions for energy and pressure are as follows,

$$\rho = \lambda T^{k+5}, \quad p = \frac{\lambda}{k+4} T^{k+5}, \tag{1.7}$$

where λ is a certain constant. Consequently, for $k \approx 3$ [8]

$$c^2 = \frac{1}{k+4} \approx 0.14. \tag{1.8}$$

This numerical value of c^2, although less than the ideal gas value $c^2 = 1/3$, is obtained very approximately due to replacement of the sum over the resonances by the integral (1.4). Strange and baryonic resonances may only occur in pairs, which is not taken into account in derivation of (1.8). Moreover, at temperatures $T \gtrsim m$, which is the case we are interested in, quantum field theory requires the inclusion of antiparticles which becomes important and therefore should be taken into account.

In 1975, Zhirov and Shuryak [11] used the Beth-Uhlenbeck method [18], which reduces the problem in the case of a purely resonant interaction to a simple case of a gas consisting of a mixture of stable and unstable particle (resonances) treated on an equal footing. Their calculations give the value of the velocity of sound $c^2 = 0.18 - 0.21$, for the range of temperatures $0.2 - 1.0$ GeV, which is appreciably smaller

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than than the asymptotic value $c^2 = 1/3$ and larger than the estimate $c^2 \approx 0.14$ of ref. [8].

In his book [4] of 1988, Shuryak proposed the "realistic" equation of state (referred to also in [1])

$$p \simeq \left(20\text{GeV}^{-2}\right) T^6, \tag{1.9}$$

describing the behavior of hadronic matter in the temperature range $0.2\text{GeV} \overset{<}{\sim} T \overset{<}{\sim} 1.0\text{GeV}$ (which gives $c^2 = 0.20$), in good agreement with the relevant experimental data.

A similar equation was obtained in refs. [22],[23] within the concept of hadronic-plasma phase transitions, for a kind of chiral bag model for the quark-gluon plasma cluster.

In ref. [12] Shuryak, in order to describe the behavior of the energy density with temperature $\rho \propto T^6$, proposed to consider hot hadronic matter as a gas of quasi-particles, which have quantum numbers of the original mesons, but with dispersion relations modified by the interaction with matter (similarly to Landau's idea of "rotons", to explain the growth of the energy density of liquid ^4He with temperature more rapidly than T^4). Since in such an approach hadrons are included as physical degrees of freedom of hot hadronic matter, they are assumed not to be absorbed too strongly, i.e., to be good quasiparticles. In ref. [12] an attempt was made to guess what these dispersion relations should be, in order to obtain the expected behavior of the energy density with temperature. In [14] these dispersion relations were calculated in explicit form.

In this paper we show that the same results can be obtained within the framework of a manifestly covariant relativistic statistical mechanics [24]-[27]. We shall review this framework briefly in the next section. First a possible correlation of the manifestly covariant theory with the hadronic spectrum of the Hagedorn-Frautschi form was discussed by Miller and Suhonen [28]. They remarked that an approach based on the grand canonical distribution function of ref. [24] gives results similar to the ones for the resonance gas in the case $\mu_K \simeq 0$, where μ_K is the additional mass potential of the ensemble [24]. In the present paper we study a free relativistic gas in the limiting case[2] $\frac{M}{\mu_K T} \to \infty$ (where M is an intrinsic scale parameter for the motion in space-time [29]), taking account of antiparticles. For such a particle-antiparticle system with $\mu \simeq 0$ (i.e., zero net particle charge) we find the equation of state $p, \rho \propto T^6$, which gives the value of the velocity of sound $c^2 = 0.20$. We obtain the relativistic mass distribution for this system and show that this distribution is a good candidate for fitting the hadronic resonances, in agreement with the theory of Hagedorn [9],[10].

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[2]For finite temperature this case is realized by $\mu_K \to 0$.

2 Relativistic N-body system

In the framework of a manifestly covariant relativistic statistical mechanics, the dynamical evolution of a system of N particles, for the classical case, is governed by equations of motion that are of the form of Hamilton equations for the motion of N *events* which generate the space-time trajectories (particle world lines) as functions of a continuous Poincaré-invariant parameter τ, called the historical time [29],[30]. These events are characterized by their positions $q^\mu = (t, \mathbf{q})$ and energy-momenta $p^\mu = (E, \mathbf{p})$ in an $8N$-dimensional phase-space. For the quantum case, the system is characterized by the wave function $\psi_\tau(q_1, q_2, \ldots, q_N) \in L^2(R^{4N})$, with the measure $d^4q_1 d^4q_2 \cdots d^4q_N \equiv d^{4N}q$, ($q_i \equiv q_i^\mu$; $\mu = 0, 1, 2, 3$; $i = 1, 2, \ldots, N$), describing the distribution of events, which evolves with a generalized Schrödinger equation [29]. The collection of events (called "concatenation" [31]) along each world line corresponds to a *particle,* and hence, the evolution of the state of the N-event system describes, *a posteriori,* the history in space and time of an N-particle system.

For a system of N interacting events (and hence, particles) one takes [31]

$$K = \sum_i \frac{p_i^\mu p_{i\mu}}{2M} + V(q_1, q_2, \ldots, q_N), \tag{2.1}$$

where M is a given fixed parameter (an intrinsic property of the particles), with the dimension of mass, taken to be the same for all the particles of the system. The Hamilton equations are

$$\frac{dq_i^\mu}{d\tau} = \frac{\partial K}{\partial p_{i\mu}} = \frac{p_i^\mu}{M},$$

$$\frac{dp_i^\mu}{d\tau} = -\frac{\partial K}{\partial q_{i\mu}} = -\frac{\partial V}{\partial q_{i\mu}}. \tag{2.2}$$

In the quantum theory, the generalized Schrödinger equation

$$i\frac{\partial}{\partial\tau}\psi_\tau(q_1, q_2, \ldots, q_N) = K\psi_\tau(q_1, q_2, \ldots, q_N) \tag{2.3}$$

describes the evolution of the N-body wave function $\psi_\tau(q_1, q_2, \ldots, q_N)$.

2.1 Ideal relativistic identical system

To describe an ideal gas of events obeying Bose-Einstein/Fermi-Dirac statistics in the grand canonical ensemble, we use the expression for the number of events found in [24],

$$N = \sum_{k^\mu} n_{k^\mu} = \sum_{k^\mu} \frac{1}{e^{(E - \mu + \mu_K \frac{m^2}{2M})/T} \mp 1}, \tag{2.4}$$

where μ_K is the additional mass potential [24], and $m^2 \equiv -k^2 \equiv -k^\mu k_\mu$. Replacing the sum over k^μ by an integral, one obtains for the density of events per unit space-time volume $n \equiv N/V^{(4)}$ [27],

$$n = \frac{1}{4\pi^3} \int_0^\infty \frac{m^3 \, dm \, \sinh^2 \beta \, d\beta}{e^{(m\cosh\beta - \mu + \mu_K \frac{m^2}{2M})/T} \mp 1}. \tag{2.5}$$

The integration results in the following expression for n [27]:

$$n = \frac{1}{(2\pi)^3} \frac{M^2}{\mu_K^2} T^2 \sum_{s=1}^\infty (\pm 1)^{s+1} \frac{e^{\frac{s\mu}{T}}}{s^2} \Psi(2, 2; \frac{sM}{2\mu_K T}), \tag{2.6}$$

where $\Psi(a, b; z)$ is the confluent hypergeometric function [32].

The formula (2.4) should be considered as the analytic continuation of (2.6). As a power series, (2.6) diverges for $|e^{\mu/T}| \geq 1$, i.e., for $\mu \geq 0$, while (2.4) is always valid.

In the limit $\frac{M}{\mu_K T} >> 1$ (or $T << \frac{M}{\mu_K}$), Eq. (2.6) reduces to [27]

$$n = \pm \frac{1}{2\pi^3} T^4 Li_4(e^{\pm\frac{\mu}{T}}), \tag{2.7}$$

where $Li_n(z) \equiv \sum_{k=1}^\infty \frac{z^k}{k^n}$ is the polylogarithm [33]. One can also obtain in this limit,

$$
\begin{aligned}
p &= \pm \frac{2}{\pi^3} \frac{T_{\triangle V}}{M} T^6 Li_6(\pm e^{\frac{\mu}{T}}), \\
\rho &= \pm \frac{10}{\pi^3} \frac{T_{\triangle V}}{M} T^6 Li_6(\pm e^{\frac{\mu}{T}}) = 5p, \\
N_0 &= \pm \frac{2}{\pi^3} \frac{T_{\triangle V}}{M} T^5 Li_5(\pm e^{\frac{\mu}{T}}),
\end{aligned} \tag{2.8}
$$

where p is the pressure and ρ is the density of energy of the *particle* gas, and N_0 is the density of *particles* per unit space volume; $T_{\triangle V}$ is the average passage interval in τ for the events which pass through a small (typical) four-volume $\triangle V$ in the neighborhood of the R^4-point [25].

3 Introduction of antiparticles

In establishing the theory presented in this paper we assumed that the total number of events and, therefore, particles, is a conserved quantity, so it makes sense to talk of a box of N particles. This can no longer be true at high temperatures [34]; it is well known that at temperatures $T \gtrsim m$ quantum field theory requires the inclusion of particle-antiparticle pair production, which becomes important and therefore should be taken into account. If $\bar{N}$ is the number of antiparticles, then N and $\bar{N}$ by themselves are not conserved but $N - \bar{N}$ is. Therefore, the high-temperature limit of (2.4) is not relevant in realistic physical systems.

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The introduction of antiparticles into the theory in a systematic way was made by Haber and Weldon [34] within the framework of the usual on-shell theory. They considered an ideal Bose gas with a conserved quantum number (referred to as "charge") Q, which corresponds to a quantum mechanical particle number operator commuting with the Hamiltonian $\hat{H}$. All thermodynamic quantities may be then obtained from the grand partition function $Tr \left\{\exp[-(\hat{H} - \hat{Q})/k_B T]\right\}$ considered as a function of T, V, and μ [35]. The formula for the conserved net charge reads[3] [34]

$$Q = \sum_{\mathbf{k}} \left[\frac{1}{e^{(E_k - \mu)/k_B T} - 1} - \frac{1}{e^{(E_k + \mu)/k_B T} - 1} \right]. \tag{3.1}$$

In such a formulation a boson-antiboson system is described by only one chemical potential μ; the sign of μ indicates whether particles outnumber antiparticles or vice versa. For the case of fermions one has simply to change the sign in the denominators of Eq. (3.1).

The introduction of antiparticles into the theory we are discussing here, as the events in the CPT-conjugate state, having nonnegative energy, leads, by application of the arguments of Haber and Weldon, to a change in sign of μ in the distribution function for antiparticles. A full consideration of particle-antiparticle pair production within the off-shell framework should include *anti-events* as well, i.e., events having the opposite sign of μ_K.

The full theory of anti-events is constructed in [36]. Upon τ reversal combined by charge conjugation, they are included in statistical mechanics as the usual events (with positive M), having the opposite sign of μ_K in the distribution function [36],[37]. The following relation which generalizes (2.4) is found as the analogue of the formula (3.1):

$$
\begin{aligned}
N &= N_E + \bar{N}_{\bar{E}} - N_{\bar{E}} - \bar{N}_E \\
&= N(\mu, \mu_K) + N(-\mu, -\mu_K) - N(-\mu, \mu_K) - N(\mu, -\mu_K) \\
&= \sum_{k^\mu} [n_{k^\mu}(\mu, \mu_K) + n_{k^\mu}(-\mu, -\mu_K) - n_{k^\mu}(-\mu, \mu_K) - n_{k^\mu}(\mu, -\mu_K)] \\
&= \sum_{k^\mu} \left[\frac{1}{e^{(E - \mu + \mu_K \frac{m^2}{2M})/k_B T} \mp 1} + \frac{1}{e^{(E + \mu - \mu_K \frac{m^2}{2M})/k_B T} \mp 1} \right. \\
&\quad \left. - \frac{1}{e^{(E + \mu + \mu_K \frac{m^2}{2M})/k_B T} \mp 1} - \frac{1}{e^{(E - \mu - \mu_K \frac{m^2}{2M})/k_B T} \mp 1} \right]
\end{aligned}
\tag{3.2}
$$

where N is the conserved net event "charge", N_E and $N_{\bar{E}}$ are the numbers of events with $E \geq 0$, $E < 0$, respectively, and $\bar{N}_E$, $\bar{N}_{\bar{E}}$ are the corresponding numbers of anti-events.

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[3]One uses the standard recipe according to which all additive thermodynamic quantities are reversed for antiparticles.

Replacing summation by integration, we now obtain a similar formula for the event number densities:

$$
\begin{aligned}
n &= n(\mu, \mu_K) + n(-\mu, -\mu_K) - n(-\mu, \mu_K) - n(\mu, -\mu_K) \\
&= \frac{1}{4\pi^3} \int_0^\infty m^3 dm \sinh^2 \beta d\beta \left[\frac{1}{e^{(m\cosh\beta - \mu + \mu_K \frac{m^2}{2M})/T} \mp 1} + \frac{1}{e^{(m\cosh\beta + \mu - \mu_K \frac{m^2}{2M})/T} \mp 1} \right. \\
&\qquad\qquad \left. - \frac{1}{e^{(m\cosh\beta + \mu + \mu_K \frac{m^2}{2M})/T} \mp 1} - \frac{1}{e^{(m\cosh\beta - \mu - \mu_K \frac{m^2}{2M})/T} \mp 1} \right].
\end{aligned}
\tag{3.3}
$$

The event–anti-event system possessing Bose-Einstein statistics was treated in ref. [37]. The requirement that the four $n_{k\mu}$'s in Eq. (3.2) be positive definite leads to the bounds on the mass spectrum [37]

$$
\frac{M}{|\mu_K|}\left(1 - \sqrt{1 - \frac{2|\mu\mu_K|}{M}}\right) \leq m \leq \frac{M}{|\mu_K|}\left(1 + \sqrt{1 - \frac{2|\mu\mu_K|}{M}}\right),
\tag{3.4}
$$

and to the relation

$$
|\mu\mu_K| \leq \frac{M}{2}.
\tag{3.5}
$$

The mass region (3.4) for small μ is well approximated by

$$
|\mu| \leq m \leq \frac{2M}{|\mu_K|}.
\tag{3.6}
$$

Therefore, for $|\mu| \simeq 0$ and $|\mu_K| \to 0$ we have almost the whole range of m, $(0, \infty)$. Since there is no requirement on the $n_{k\mu}$'s in the fermionic case, we take for this case $0 \leq m < \infty$. Thus, both cases can be treated simultaneously, with the whole range of m. In this way we obtain, upon expansion of the denominators in Eq. (3.3) into power series and integration on β,

$$
n = -\frac{T}{\pi^3} \int_0^\infty dm \sum_{s=1}^\infty \frac{(\pm 1)^{s+1}}{s} m^2 \sinh\frac{s\mu}{T} \sinh\frac{s\mu_K m^2}{2MT} K_1\left(\frac{sm}{T}\right),
\tag{3.7}
$$

from which we identify the mass distribution $f(m)$, normalized as[4]

$$
\left| \int_0^\infty dm f(m) \right| = |n|.
\tag{3.8}
$$

In (3.7), K_1 is a Bessel function of the third kind (imaginary argument).

Integration in (3.7) gives the following expression for n :

$$
n = -\frac{1}{4\pi^3}\frac{M^2}{\mu_K^2}T^2 \sum_{s=1}^\infty \frac{(\pm 1)^{s+1}}{s^2} \sinh\frac{s\mu}{T}\left[\Psi(2,2;\frac{sM}{2\mu_K T}) - \Psi(2,2;-\frac{sM}{2\mu_K T})\right].
\tag{3.9}
$$

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[4]We remark that only $|n|$ has physical meaning as a conserved net event charge.

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The formulas for p, ρ and N_0 become [36]

$$
\begin{align}
p &= p(\mu, \mu_K) + p(-\mu, -\mu_K) + p(-\mu, \mu_K) + p(\mu, -\mu_K), \tag{3.10} \\
\rho &= \rho(\mu, \mu_K) + \rho(-\mu, -\mu_K) + \rho(-\mu, \mu_K) + \rho(\mu, -\mu_K), \tag{3.11} \\
N_0 &= N_0(\mu, \mu_K) + N_0(\mu, -\mu_K) - N_0(-\mu, \mu_K) - N_0(-\mu, -\mu_K), \tag{3.12}
\end{align}
$$

where $p(\mu, \mu_K)$, $\rho(\mu, \mu_K)$ and $N_0(\mu, \mu_K)$ are the subdistributions defined by the corresponding formulas of the event statistical mechanics [27].

Note that the expressions (3.12) for N_0 (actually corresponding to the Q in Eq. (3.1)), which refers to particles and antiparticles, and (3.3) for n, which refers to *events*, differ in sign of the corresponding terms containing $-\mu_K$. While N_0 commutes with the effective Hamiltonian of a relativistic many body system in the usual framework, it is (3.2) which commutes with the evolution operator of the covariant theory [36].

We also remark that, as in the usual theory, the system we are studying is described by only one chemical potential μ and only one mass potential μ_K; the sign of μ indicates whether particles outnumber antiparticles or vice versa, while the sign of $\mu\mu_K$ has a similar relation to the relative number of events and anti-events, as seen in Eqs. (4.8).

4 Realistic equation of state for event–anti-event system

In the limit $\frac{M}{|\mu_K|} >> T$, we use the asymptotic formula for $z \to \infty$ (ref. [32], p.278, subsection 6.13.1)

$$
\Psi(a, a; z) \sim z^{-a} \left(1 - \frac{a}{z} + \frac{a(a+1)}{z^2} - \frac{a(a+1)(a+2)}{z^3} + \dots \right), \tag{4.1}
$$

and obtain from (3.9), retaining the terms up to the order of $(\frac{\mu_K T}{M})^2$,

$$
\begin{align}
n &= \mp \frac{4}{\pi^3} \frac{\mu_K}{M} T^5 \left\{ \left[Li_5(\pm e^{\frac{\mu}{T}}) - Li_5(\pm e^{-\frac{\mu}{T}}) \right] \right. \notag \\
&\quad \left. + 48 \frac{\mu_K^2 T^2}{M^2} \left[Li_7(\pm e^{\frac{\mu}{T}}) - Li_7(\pm e^{-\frac{\mu}{T}}) \right] \right\}. \tag{4.2}
\end{align}
$$

$$\tag{4.3}$$

The corresponding formulas for p, ρ and N_0 read [27]

$$
p = \pm \frac{4}{\pi^3} \frac{T_{\triangle V}}{M} T^6 \left\{ \left[Li_6(\pm e^{\frac{\mu}{T}}) + Li_6(\pm e^{-\frac{\mu}{T}}) \right] \right.
$$

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$$+48\frac{\mu_K^2 T^2}{M^2}\left[Li_8(\pm e^{\frac{\mu}{T}}) + Li_8(\pm e^{-\frac{\mu}{T}})\right]\Bigg\}, \tag{4.4}$$

$$\rho = \pm\frac{4}{\pi^3}\frac{T_{\triangle V}}{M}T^2\left\{5\left[Li_6(\pm e^{\frac{\mu}{T}}) + Li_6(\pm e^{-\frac{\mu}{T}})\right]\right.$$
$$\left. +432\frac{\mu_K^2 T^2}{M^2}\left[Li_8(\pm e^{\frac{\mu}{T}}) + Li_8(\pm e^{-\frac{\mu}{T}})\right]\right\}, \tag{4.5}$$

$$N_0 = \pm\frac{4}{\pi^3}\frac{T_{\triangle V}}{M}T^5\left\{\left[Li_5(\pm e^{\frac{\mu}{T}}) - Li_5(\pm e^{-\frac{\mu}{T}})\right]\right.$$
$$\left. +48\frac{\mu_K^2 T^2}{M^2}\left[Li_7(\pm e^{\frac{\mu}{T}}) - Li_7(\pm e^{-\frac{\mu}{T}})\right]\right\}. \tag{4.6}$$

Note the remarkable relation
$$\left|N_0\right| = \frac{T_{\triangle V}}{|\mu_K|}|n|. \tag{4.7}$$

First consider the bosonic case. Using the corresponding relations for the polylogarithms (see Appendix A), and retaining the terms up to the order of $(\frac{\mu_K T}{M})^2$, $(\frac{\mu}{T})^2$, we obtain

$$n = -\frac{4\pi}{45}\frac{\mu\mu_K}{M}T^4\left(1 + \frac{5}{2\pi^2}\frac{\mu^2}{T^2} + \frac{96\pi^2}{21}\frac{\mu_K^2 T^2}{M^2}\right),$$

$$p = \frac{8\pi^3}{945}\frac{T_{\triangle V}}{M}T^6\left(1 + \frac{21}{4\pi^2}\frac{\mu^2}{T^2} + \frac{24\pi^2}{5}\frac{\mu_K^2 T^2}{M^2}\right),$$

$$\rho = \frac{40\pi^3}{945}\frac{T_{\triangle V}}{M}T^6\left(1 + \frac{21}{4\pi^2}\frac{\mu^2}{T^2} + \frac{216\pi^2}{25}\frac{\mu_K^2 T^2}{M^2}\right),$$

$$N_0 = \frac{4\pi}{45}\frac{T_{\triangle V}}{M}\mu T^4\left(1 + \frac{5}{2\pi^2}\frac{\mu^2}{T^2} + \frac{96\pi^2}{21}\frac{\mu_K^2 T^2}{M^2}\right). \tag{4.8}$$

Therefore,
$$c^2 = \frac{dp}{d\rho} \simeq \frac{p}{\rho} \cong \frac{1}{5}\left(1 - \frac{96}{25}\frac{\mu_K^2 T^2}{M^2}\right), \tag{4.9}$$

i.e., one obtains the value of the velocity of sound $c^2 = 0.20$, as well as the correction to this value, of the order of $(\frac{\mu_K T}{M})^2$.

In the fermionic case, one uses the corresponding relations for the polylogarithms with the argument $z < 0$ (Appendix A), and obtains similar formulas for n, p, ρ and N_0, which give

$$c^2 \cong \frac{1}{5}\left(1 - \frac{127}{124}\frac{96}{25}\frac{\mu_K^2(k_B T)^2}{M^2}\right).$$

We see that in the case $\mu \simeq 0$, $\mu_K \to 0$ the equation of state for event–anti-event system reads $p, \rho \propto T^6$, implying the velocity of sound $c^2 = 0.20$, in agreement with the Shuryak's "realistic" equation of state. We now turn to the corresponding distribution of mass.

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5 Relativistic mass distribution in event–anti-event system

The distribution of mass, defined by Eq. (3.7), can be simplified in the case $\mu \simeq 0$, $\frac{M}{|\mu_K|T} >> 1$. Since for the whole range of m and small μ the average $\langle m \rangle$ depends only on temperature [26],[27], $\frac{|\mu_K|m^2}{2MT}$ is of the order of $\frac{|\mu_K|T}{M} << 1$. Therefore, one can replace the hyperbolic sine functions by their arguments and obtain in this way

$$f(m) \simeq \frac{1}{2\pi^3}\frac{|\mu\mu_K|}{MT}\sum_{s=1}^{\infty}(\pm 1)^{s+1}sm^4 K_1\left(\frac{sm}{T}\right). \tag{5.1}$$

For a given T, in the region $m << T$ we use the asymptotic formula (ref. [38], p.375)

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad z \to 0,$$

and obtain

$$f(m) \sim m^3, \tag{5.2}$$

for both bosonic and fermionic cases. If we assume that the distribution (5.2) describes the genuine spectrum of compound hadron-resonance systems, and restrict ourselves only to the region $m << T$, we shall obtain, with Shuryak's Eqs. (1.3),(1.8) for $k = 3$,

$$c^2 = 0.14.$$

In the region $m >> T$, we use another asymptotic formula (ref. [38], p. 378)

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}, \quad z \to \infty.$$

Since in this region particles become distinguishable, we have

$$f(m) \sim m^{7/2}e^{-\frac{m}{T}}. \tag{5.3}$$

In this case the equation of state tends asymptotically to the equation of state of an ideal Stefan-Boltzmann gas, $p = \rho/3$, providing for the velocity of sound the value

$$c^2 = 0.33.$$

For the transitional region $m \stackrel{<}{\sim} T$, the formula for $K_\nu(z)$ can be found by the method of steepest descents (ref. [32], Vol.2, p.28). The corresponding expression for the mass distribution can be also obtained directly from Eq. (5.3), since in this case the power function dominates over the exponent. In either case, one has

$$f(m) \sim m^{7/2}, \tag{5.4}$$

so that, if we restrict ourselves only to this region, we shall obtain, through Eqs.
(1.3),(1.8) with $k = 7/2$,

$$c^2 = \frac{2}{15} \approx 0.13.$$

If the three cases are treated on an equal footing, one has

$$c^2 = \frac{1}{3}(0.14 + 0.13 + 0.33) = 0.20, \tag{5.5}$$

in agreement with the equation of state (4.9) for the whole mass region. We see that
the mass distribution (5.1) is a good candidate for fitting the hadronic resonances, in
good agreement with the phenomenological models of Hagedorn, Shuryak, *et al.*

5.1 Specific heat of event–anti-event gas

In conclusion we wish to calculate the specific heat of the event–anti-event gas we are
studying. The distribution (5.1) gives the following value of the average m :

$$\langle m \rangle = \frac{\int dm\, m f(m)}{\int dm f(m)} = \frac{\int_0^\infty dm\, m^5 K_1(\frac{m}{T})}{\int_0^\infty dm\, m^4 K_1(\frac{m}{T})} = \frac{45\pi}{32} T, \tag{5.6}$$

where we used the formula (ref. [39], p.684, formula 16)

$$\int_0^\infty dx\, x^\mu K_\nu(ax) = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{5.7}$$

The rest frame average of the energy $\langle m \cosh \beta \rangle$ can be obtained from (3.3),(5.1),
with the help of the relation

$$\sinh^2 \beta \cosh \beta = \frac{1}{4}(\cosh 3\beta - \cosh \beta)$$

and the formula (ref. [39], p.358, formula 4)

$$\int_0^\infty dx \cosh \nu x\, e^{-a \cosh x} = K_\nu(a), \tag{5.8}$$

as follows,

$$
\begin{aligned}
\langle E \rangle &= \frac{1}{n} \int m^3 dm \sinh^2 \beta d\beta \left[\frac{m \cosh \beta}{e^{(m \cosh \beta - \mu + \mu_K \frac{m^2}{2M})/T} \mp 1} + \text{ three terms} \right] \\
&= \frac{1}{4T} \frac{\int_0^\infty dm\, m^6 \left[K_3(\frac{m}{T}) - K_1(\frac{m}{T}) \right]}{\int_0^\infty dm\, m^4 K_1(\frac{m}{T})} = 6T.
\end{aligned}
\tag{5.9}
$$

This value is twice as much as Pauli's classical value $3T$ obtained in the ultrarelativis-
tic limit within the usual on-shell theory [40], which is quite natural, since we take

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account of particle-antiparticle pairs. The value of the specific heat is now obtained from (5.6),(5.9):

$$\langle E - m \rangle = \left(6 - \frac{45\pi}{32} \right) T = \gamma' \frac{3}{2} T, \tag{5.10}$$

$$\gamma' = 4 - \frac{15\pi}{16} \approx 1.055. \tag{5.11}$$

We see that at low temperatures, γ' provides a correction of the order of 5.5% to the classical value 3/2.

6 Concluding remarks

We have discussed possible consequences of a manifestly covariant relativistic statistical mechanics for hadronic physics. We have considered the relativistic event–anti-event system, for which we have obtained the relativistic mass distribution and calculated all characteristic thermodynamic variables. We have shown that in the case of small μ_K and zero net particle charge, the equation of state for such a system corresponds to the "realistic" equation of state proposed by Shuryak [4] to describe the behavior of hot hadronic matter. We have seen that the mass distribution obtained in this paper can serve as a framework for fitting the hadronic resonances, in good agreement with the phenomenological models of Hagedorn *et al* [8]-[10].

In ref. [41] we have argued that the mass potential μ_K is related to the degeneracy parameter δ : $\mu_K \sim \delta^{-a}$, $a > 0$. Therefore, small μ_K indicates large δ, i.e., high degeneracy. In this case the system should exhibit the collective behavior determined by a relatively strong interaction, mainly of attractive character. This attraction considerably reduces the pressure, resulting in a value for the sound velocity $c^2 \approx 0.20$.

Further physical consequences of the theory of the off-shell events are discussed in refs. [37],[41]. In [37] the statistical mechanics of the bosonic event–anti-event system is considered. For such a system, at some critical temperature a special type of Bose-Einstein condensation sets in, which provides the events making up the ensemble a definite mass and represents, in this way, a phase transition to an equilibrium on-shell sector. In [12] an adiabatic equation of state, $p \propto N_0^{6/5}$, is obtained for the system of degenerate off-shell fermions and possible implications in astrophysics are discussed.

The various aspects of the theory of the off-shell events discussed in this paper, as well as related questions, are now under further investigation.

Appendix A

The polylogarithm is defined by the relation

$$Li_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}, \quad |z| < 1 \text{ or } |z| = 1, \ Re\ \nu > 1. \tag{A.1}$$

For $|z| \geq 1$, the function $Li_\nu(z)$ is defined as the analytic continuation of this series. For integer n and positive z, the following relation holds (ref. [33], p.763):

$$Li_n(z) + (-1)^n Li_n\left(\frac{1}{z}\right) = -\frac{(2\pi i)^n}{n!} B_n\left(\frac{\ln z}{2\pi i}\right), \tag{A.2}$$

where B_n are the Bernoulli polynomials (ref. [33], p.765),

$$B_0(z) = 1, \quad B_1(z) = z - \frac{1}{2}, \quad B_2(z) = z^2 - z + \frac{1}{6},$$

$$B_3(z) = z^3 - \frac{3}{2}z^2 + \frac{1}{2}z, \quad B_4(z) = z^4 - 2z^3 + z^2 - \frac{1}{30}, \quad \text{etc.}$$

possessing the property

$$B_n'(z) = nB_{n-1}(z).$$

In view of (A.2) we have

$$Li_1(z) - Li_1\left(\frac{1}{z}\right) = -\ln z + \pi i, \tag{A.3}$$

$$Li_2(z) + Li_2\left(\frac{1}{z}\right) = -\frac{\ln^2 z}{2} + \pi i \ln z + \frac{\pi^2}{3}, \tag{A.4}$$

$$Li_3(z) - Li_3\left(\frac{1}{z}\right) = -\frac{\ln^3 z}{6} + \frac{\pi i}{2}\ln^2 z + \frac{\pi^2}{3}\ln z, \tag{A.5}$$

$$Li_4(z) + Li_4\left(\frac{1}{z}\right) = -\frac{\ln^4 z}{24} + \frac{\pi i}{6}\ln^3 z + \frac{\pi^2}{6}\ln^2 z + \frac{\pi^4}{45}, \tag{A.6}$$

$$Li_5(z) - Li_5\left(\frac{1}{z}\right) = -\frac{\ln^5 z}{120} + \frac{\pi i}{24}\ln^4 z + \frac{\pi^2}{18}\ln^3 z + \frac{\pi^4}{45}\ln z, \tag{A.7}$$

$$Li_6(z) + Li_6\left(\frac{1}{z}\right) = -\frac{\ln^6 z}{720} + \frac{\pi i}{120}\ln^5 z + \frac{\pi^2}{72}\ln^4 z + \frac{\pi^4}{90}\ln^2 z + \frac{2\pi^6}{945}, \quad \text{etc.} \tag{A.8}$$

For integer n and negative z, we use another relation (ref. [33], p.763):

$$Li_n(z) + (-1)^n Li_n\left(\frac{1}{z}\right) = -\sum_{k=1}^{[n/2]} c_{k,n} \ln^{n-2k}(-z); \qquad (A.9)$$

$$c_{0,n} = \frac{1}{n!}, \quad c_{1,n} = \frac{\pi^2}{6(n-2)!},$$

$$c_{k,n} = (-1)^{k+1}\frac{2(2^{2k-1}-1)\pi^{2k}B_{2k}}{(2k)!(n-2k)!}, \quad k = 0, 1, \ldots, [n/2],$$

where B_{2k} are the Bernoulli numbers,

$$B_0 = 1, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \text{etc.}$$

In this way we obtain

$$Li_1(z) - Li_1\left(\frac{1}{z}\right) = -\ln(-z), \qquad (A.10)$$

$$Li_2(z) + Li_2\left(\frac{1}{z}\right) = -\left[\frac{\ln^2(-z)}{2} + \frac{\pi^2}{6}\right], \qquad (A.11)$$

$$Li_3(z) - Li_3\left(\frac{1}{z}\right) = -\left[\frac{\ln^3(-z)}{6} + \frac{\pi^2}{6}\ln(-z)\right], \qquad (A.12)$$

$$Li_4(z) + Li_4\left(\frac{1}{z}\right) = -\left[\frac{\ln^4(-z)}{24} + \frac{\pi^2}{12}\ln^2(-z) + \frac{7\pi^4}{360}\right], \qquad (A.13)$$

$$Li_5(z) - Li_5\left(\frac{1}{z}\right) = -\left[\frac{\ln^5(-z)}{120} + \frac{\pi^2}{36}\ln^3(-z) + \frac{7\pi^4}{360}\ln(-z)\right], \qquad (A.14)$$

$$Li_6(z) + Li_6\left(\frac{1}{z}\right) = -\left[\frac{\ln^6(-z)}{720} + \frac{\pi^2}{144}\ln^4(-z) + \frac{7\pi^4}{720}\ln^2(-z) + \frac{31\pi^6}{15120}\right] \qquad (A.15)$$

etc.

The formulas in the main text are obtained from (A.3)-(A.8) for $z = e^{\frac{\mu}{T}}$, and from (A.10)-(A.15) for $z = -e^{\frac{\mu}{T}}$, for the bosonic and the fermionic cases, respectively.

References

[1] E.V. Shuryak, Phys. Repts. **61** (1980) 72

[2] R.D. Pisarsky, D.J. Gross and L.G. Yaffe, Rev. Mod. Phys. **53** (1981) 43

[3] B. Müller, The Physics of the Quark-Gluon Plasma, (Springer-Verlag, Berlin, 1985)

[4] E.V. Shuryak, The QCD Vacuum, Hadrons and the Superdense Matter, (World Scientific, Singapore, 1988)

[5] J. Cleymans, R.V. Gavai and E. Suhonen, Phys. Repts. **130** (1986) 217

[6] L.D. Landau, Izv. Akad. Nauk SSSR, seriya Fiz. **17** (1953) 51

[7] G.A. Melekhin, JETP, **8** (1959) 829

[8] E.V. Shuryak, Sov. J. Nucl. Phys. **16** (1973) 220

[9] R. Hagedorn, Nuovo Cimento **35** (1965) 216; Suppl. Nuovo Cimento **3** (1965) 147; **6** (1968) 311

[10] S.C. Frautschi, Phys. Rev. D **3** (1971) 2821

[11] O.V. Zhirov and E.V. Shuryak, Sov. J. Nucl. Phys. **21** (1975) 443

[12] E.V. Shuryak, Phys. Rev. D **42** (1990) 1764

[13] E.V. Shuryak, Nucl Phys. A **522** (1991) 377c; **533** (1991) 761

[14] E.V. Shuryak, Nucl Phys. A **536** (1992) 739

[15] S. Weinberg, Phys. Rev. Lett. **17** (1966) 616

[16] P. Gerber and H. Leutwyler, Nucl. Phys. B **321** (1989) 387

[17] S.Z. Belenky and L.D. Landau, Sov. Phys. Uspekhi **62** (1956) 1

[18] E. Beth and G.E. Uhlenbeck, Physica **3** (1936) 729

[19] L.D. Landau and E.M. Lifshitz, Statistical Physics, Part 1, (Pergamon Press, Oxford, 1986), p.236

[20] R.D. Carlitz, Phys. Rev. D **5** (1972) 3231

[21] G.A. Melekhin, Proc. of International Conference on Cosmic Rays, (Nauka, Moscow, 1960)

[22] A. Schnabel and J. Rafelski, Diploma thesis, Frankfurt, 1984

[23] B. Muller and J.M. Eisenberg, Nucl. Phys. A **435** (1985) 791

[24] L.P. Horwitz, W.C. Schieve and C. Piron, Ann. Phys. (N.Y.) **137** (1981) 306

[25] L.P. Horwitz, S. Shashoua and W.C. Schieve, Physica A **161** (1989) 300

[26] L. Burakovsky and L.P. Horwitz, Physica A **201** (1993) 666

[27] L. Burakovsky and L.P. Horwitz, Equilibrium Relativistic Mass Distribution for Indistinguishable Events, Tel-Aviv Univ. Preprint TAUP-2081-93

[28] D.E. Miller and E. Suhonen, Phys. Rev. D**26** (1982) 2944

[29] L.P. Horwitz and C. Piron, Helv. Phys. Acta **46** (1973) 316

[30] E.C.G. Stueckelberg, Helv. Phys. Acta **14** (1941) 372, 588; **15** (1942) 23

[31] R. Arshansky, L.P. Horwitz and Y. Lavie, Found. Phys. **13** (1983) 1167

[32] A. Erdélyi, ed., Higher Transcendental Functions, Bateman Series, (McGraw-Hill, New York, 1953), Vol. 1, p.257, section 6.6

[33] A.P. Prudnikov *et al.* Integrals and series, (Gordon and Breach, New York, 1980), Vol. 3, p.762, Appendix II.5

[34] H.E. Haber and H.A. Weldon, Phys. Rev. Lett. **46** (1981) 1497

[35] K. Huang, Statistical Mechanics, (Wiley, New York, 1963)

[36] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Statistical Mechanics and Thermodynamics of Event–Anti-event System, *in preparation*

[37] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Relativistic Bose-Einstein Condensation to Mass Shell Limit, Tel-Aviv Univ. Preprint TAUP-2149-94

[38] M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions, (Dover, New York, 1972)

[39] I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Series, and Products, (Academic Press, New York, 1980)

[40] W. Pauli, The Theory of Relativity, (Pergamon Press, London, 1958)

[41] L. Burakovsky, L.P. Horwitz and W.C. Schieve, Statistical Mechanics of Relativistic Degenerate Fermi Gas I.Cold Adiabatic Equation of State, Tel-Aviv Univ. Preprint TAUP-2124-93